

- 1 The matrix that will reflect the plane in the y -axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will dilate the result by a factor of 3 from the x -axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

- 2 The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will reflect the result in the x -axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

- 3 a The matrix that will reflect the plane in the x -axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrix that will reflect the plane in the y -axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- b The matrix that will rotate the plane by 180° clockwise is given by

$$\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the same as the matrix found above.

- 4 a T_1 : The matrix that will reflect the plane in the x -axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

T_2 : The matrix that will dilate the result by a factor of 2 from the y -axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

c No. The order of transformation does not matter in this instance, since the two matrices are the same.

5 a T_1 : The matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \\ \sin(-90^\circ) & \cos(-90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

T_2 : The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c Yes. The order of transformation does matter in this instance, as the two matrices are different (the first gives the reflection in the y -axis, the second a reflection in the x -axis).

6 a

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -x - 3 \\ y + 5 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \rightarrow (-x - 3, y + 5)$.

b

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - 3 \\ y + 5 \end{bmatrix} \\ &= \begin{bmatrix} -x + 3 \\ y + 5 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \rightarrow (-x + 3, y + 5)$.

c Yes. The order of transformation does matter in this instance, as the rule for each composition is different.

7 a This is a reflection in the x -axis followed by a dilation from the y -axis by a factor of 2 (or visa versa):

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b This is a reflection in the x -axis followed by a dilation from the x -axis by a factor of 3 (or visa versa):

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.$$

c This is a reflection in the line $y = x$ followed by a dilation from the x -axis by a factor of 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Alternatively, it is a dilation from the y -axis by a factor of 2 followed by a reflection in the line $y = x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

d This is a reflection in the line $y = -x$ followed by a dilation from the y -axis by a factor of 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

Alternatively, it is a dilation from the x -axis by a factor of 2 followed by a reflection in the line $y = -x$:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

8 a The required matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

b The above matrix corresponds to a rotation by angle $\theta = 90^\circ$ since

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

9 We require that:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
$$\begin{bmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix}$$

For these two matrices to be equal, we required that

$$-\sin \theta = \sin \theta$$

$$2 \sin \theta = 0$$

$$\sin \theta = 0$$

$$\theta = 180^\circ k, \text{ where } k \in \mathbb{Z}.$$

10a Matrix A^2 will rotate the plane by angle 2θ .

b

$$A^2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$$

c Since A^2 will rotate the plane by angle 2θ , another expression for A^2 is

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Equating the two expressions for A^2 gives

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}.$$

Therefore,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

11a

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} y + 1 \\ x + 2 \end{bmatrix}$$

Therefore, the rule can be written in the form $(x, y) \rightarrow (y + 1, x + 2)$ or in the form $\begin{matrix} x' = y + 1 \\ y' = x + 2. \end{matrix}$

b We have,

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} y + 1 \\ x + 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x + 2 \\ y + 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

This shows the transformation can be expressed as a reflection in the line $y = x$ followed by a translation in the direction of vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

12a

$$\begin{aligned} &\begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} &\begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \end{aligned}$$

c A 60° rotation followed by a -45° rotation will give a 15° rotation. Therefore, the required matrix is.

$$\begin{aligned} &\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2} + \sqrt{6}}{4} & \frac{\sqrt{2} - \sqrt{6}}{4} \\ \frac{\sqrt{6} - \sqrt{2}}{4} & \frac{\sqrt{6} + \sqrt{2}}{2} \end{bmatrix} \end{aligned}$$

The rotation matrix of 15° is also given by the expression

$$\begin{bmatrix} \cos(15) & -\sin(15) \\ \sin(15) & \cos(15) \end{bmatrix}.$$

d Comparing the entries of these two matrices gives

$$\cos 15^\circ = \frac{\sqrt{2} + \sqrt{6}}{4},$$

$$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4},$$

13 The matrix that will reflect the plane in the line $y = x \tan \phi$ is

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = x \tan \theta$ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{aligned} & \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \\ = & \begin{bmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & \cos 2\theta \sin 2\phi - \sin 2\theta \cos 2\phi \\ \sin 2\theta \cos 2\phi - \cos 2\theta \sin 2\phi & \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi \end{bmatrix} \\ = & \begin{bmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & -(\cos 2\theta \sin 2\phi - \cos 2\theta \sin 2\phi) \\ \cos 2\theta \sin 2\phi - \cos 2\theta \sin 2\phi & \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi \end{bmatrix} \\ = & \begin{bmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi) \end{bmatrix} \end{aligned}$$

This is a rotation matrix corresponding to angle $2\theta - 2\phi$.